String Theory 250
Problem Set #2
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Heat Kernel Techniques

1 De Witt-Seeley coefficients

Find the expressions $A(f)$ and $C(f)$ in the De Witt-Seeley expansion

$$\mathrm{Tr}\,\left(fe^{t\nabla^2}\right) \equiv \int d^2\sigma\sqrt{\gamma(\sigma)}\,f(\sigma)\langle\sigma|e^{t\nabla^2}|\sigma\rangle = \frac{1}{t}A(f) + C(f) + O(t),$$

where $f(\sigma) \equiv f(\sigma^1, \sigma^2)$ is a continuous function on the worldsheet (differentiable as many times as you need), $\gamma \equiv \det\gamma_{ab}$ is the determinant of the worldsheet metric, and

$$\nabla^2 \equiv \frac{1}{\sqrt{\gamma}}\partial_a\left(\sqrt{\gamma}\gamma^{ab}\partial_b\right)$$

is the Laplacian operator.

Hint: You know that $A(f)$ and $C(f)$ are generally covariant local expressions of the form

$$\int\sqrt{\gamma(\sigma)}\Phi(\sigma)f(\sigma)d^2\sigma,$$

where $\Phi(\sigma)$ depends on the metric. To find $\Phi(0)$, use a Diff transformation to bring the metric to the form

$$\gamma_{ab} = \exp(2\phi(\sigma))\delta_{ab}$$

such that $\phi(0) = \partial_a\phi(0) = 0$. (See §11.A of [1].)

2 Chiral anomaly in $D = 4$

The heat-kernel technique can also be applied to the chiral anomaly of fermions in $D = 4$. Take a massless Dirac fermion field ψ with charge q, so that its Lagrangian is

$$L = i\overline{\psi}\gamma^\mu D_\mu\psi, \qquad D_\mu \equiv \partial_\mu - iqA_\mu,$$

where A_μ is a given external gauge field. Use the heat-kernel technique to find the anomaly under a chiral transformation

$$\psi \to e^{i\epsilon\gamma^5}\psi.$$

We are looking for a formula for the variation of the measure in the Feynman path integral:

$$[\mathcal{D}\psi][\mathcal{D}\overline{\psi}] \to [\mathcal{D}\psi][\mathcal{D}\overline{\psi}]\exp(i\epsilon W).$$

Show that

$$W = \mathrm{Tr}\,\left(\gamma^5 e^{tD^2}\right), \qquad \text{where} \quad D^2 \equiv (\gamma^\mu D_\mu)^2,$$

and use the heat-kernel techniques to express W in terms of the field strength $F_{\mu\nu}$.

References

[1] M. B. Green, J. H. Schwarz and E. Witten, “Superstring Theory,” Cambridge University Press.