Optimal Energy Weighting for X-ray Detection of Small Objects

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1 Optimal Setting of a Monochromatic Source

Suppose we have an x-ray source that can be set to any desired single energy and that we want to have maximal sensitivity for detecting a small object with high density, and x-ray cross-section σ_T. We suppose it is embedded in a uniform body with cross-section σ_B. The depths in atoms/unit area are L_T and L_B and we write

$$X_T = \sigma_T L_T; \qquad X_B = \sigma_B L_B. \tag{1}$$

If N x-rays enter the body in the absence of the small target, the number that emerge is Ne^{-X_B}. If the target is present, as well, this is reduced to $Ne^{-X_B+X_T} \approx Ne^{-X_B}(1 - X_T)$. We assume $X_T \ll 1$.

The change in the signal due to the target is, on average,

$$\Delta N = Ne^{-X_B} X_T \tag{2}$$

The statistical significance of this is

$$\chi^2 = \frac{(\Delta N)^2}{\sigma_N^2} = \frac{N^2 e^{-2X_B} X_T^2}{Ne^{-X_B}} = Ne^{-X_B} X_T^2 \tag{3}$$

We wish to maximize this consistent with a fixed exposure of the body (patient) the x-rays. The exposure is the number of x rays absorbed times their energy:

$$\text{Exposure} = EN(1 - e^{-X_B}) \tag{4}$$

Thus we need to maximize

$$\frac{e^{-X_B} X_T^2}{E(1 - e^{-X_B})} \tag{5}$$

where, of course, the cross-sections and thus X_B and X_T are functions of the energy. The solution is found by setting the derivative of χ^2 with respect to energy to zero. It is a good approximation to take the x-ray cross sections to be proportional to E^{-3}. It then suffices to maximize

$$\frac{{X_B}^{7/3}}{e^{X_B}-1} \tag{6}$$

This leads to the transcendental equation

$$1-e^{-X_B}=\frac{3}{7}X_B \tag{7}$$

whose solution is

$$X_B=2.03; \qquad 1-e^{-X_B}=0.87 \tag{8}$$

That is, one should choose the energy so that 87% is absorbed by the medium.

2 Energy Weighting with a given Energy Spectrum

Suppose there is a fixed x-ray source with a spectrum (number of x rays per unit photon energy) $\phi(E)$. Imagine that we bin the x-ray energies into M bins each of width ΔE. The $\phi(E_i)\Delta E$ plays the role of N in the calculation above. Now suppose we weight each bin by w_i and form the single statistic

$$W=\sum_i n_i w_i \tag{9}$$

where n_i is the number of x rays observed in a detector (assumed to be perfectly efficient) in the ith energy bin. The expectation value of n_i when there is no target present is

$$\langle n_i\rangle=\phi(E_i)\Delta E e^{-X_B} \tag{10}$$

where X_B is evaluated at E_i. Analogously, when the target is present the result is

$$\langle n_i\rangle'=\phi(E_i)\Delta E e^{-X_B}(1-X_T) \tag{11}$$

Thus, writing $X_{Ti} = X_T(E_i)$

$$\langle \Delta W \rangle = \sum_i \langle n_i \rangle X_{Ti} w_i \tag{12}$$

To judge the significance of this signal we need to know the fluctuations in W (we can ignore the target here).

$$\langle W^2 \rangle - \langle W \rangle^2 = \langle \sum_i n_i w_i \sum_j n_j w_j \rangle - (\langle \sum_i n_i w_i \rangle)^2 \tag{13}$$

Using Poisson statistics, $\langle n_i^2 \rangle - \langle n_i \rangle^2 = \langle n_i \rangle$, so

$$\sigma_W^2 = \langle W^2 \rangle - \langle W \rangle^2 = \sum_i \langle n_i \rangle w_i^2 \tag{14}$$

Altogether, our significance is

$$S = \frac{\langle \Delta W \rangle^2}{\sigma_W^2} = \frac{(\sum_i \langle n_i \rangle X_{Ti} w_i)^2}{\sum_i \langle n_i \rangle w_i^2} \tag{15}$$

Let us define

$$\bar{S} = \frac{\langle \Delta W \rangle^2}{\sigma_W^2 \sum_j \langle n_j \rangle X_{Tj}^2} = \frac{(\sum_i \langle n_i \rangle X_{Ti} w_i)^2}{\sum_i \langle n_i \rangle w_i^2 \sum_j \langle n_j \rangle X_{Tj}^2} \tag{16}$$

We recognize that $\sum_i a_i b_i$ defines a scalar product and that $\bar{S}$ is simply $\cos^2\theta$ where θ is the angle between the abstract vectors w_i and X_{Ti}. Since S and $\bar{S}$ differ only by a factor independent of w_i, is suffices to maximize the latter. To do this we need to choose the vectors to be parallel to each other, i.e.

$$w_i \propto X_{Ti} \propto \sigma_T(E_i) \tag{17}$$

Now if the x ray cross-section varies as E^{-3}, the ideal weighting is to take w_i to be proportional to E^{-3} as well.

As an example, consider a spectrum extending between $E_{min} = 15$ keV and $E_{max} \propto 30$ keV:

$$\phi \propto (E_{max} - E)(E - E_{min}) \tag{18}$$

	$\cos^2\theta$
$w \propto E$	0.784
$w = 1$	0.866
$w = 1 - \log(E/E_{min})/\log(E_{max}/E_{min})$	0.970
$w \propto \sigma_T$	1.0

Table 1: Values of the figure of merit, $\cos^2\theta$ for various weightings. The "body" is 5 cm of $Z = 7.0$ material. The target has $Z = 20$.

Take the "body" material to have an effective Z of 7 and take the target to be calcium ($Z = 20$). Use our parameterization

$$\sigma = 24.15 Z^{4.2} E^{-3} + 0.56 Z \tag{19}$$

which gives the cross-section in barns, with E measured in keV. Consider a sample 5 cm in depth.

We consider four different weightings:

- $w \propto E$
- $w = 1$
- $w = 1 - \log(E/E_{min})/\log(E_{max}/E_{min})$
- $w \propto \sigma_T$

The third is chosen so the weight drops from one to zero across the spectrum.

We then calculate $\cos^2\theta$ as described above. The results are given in the Table.

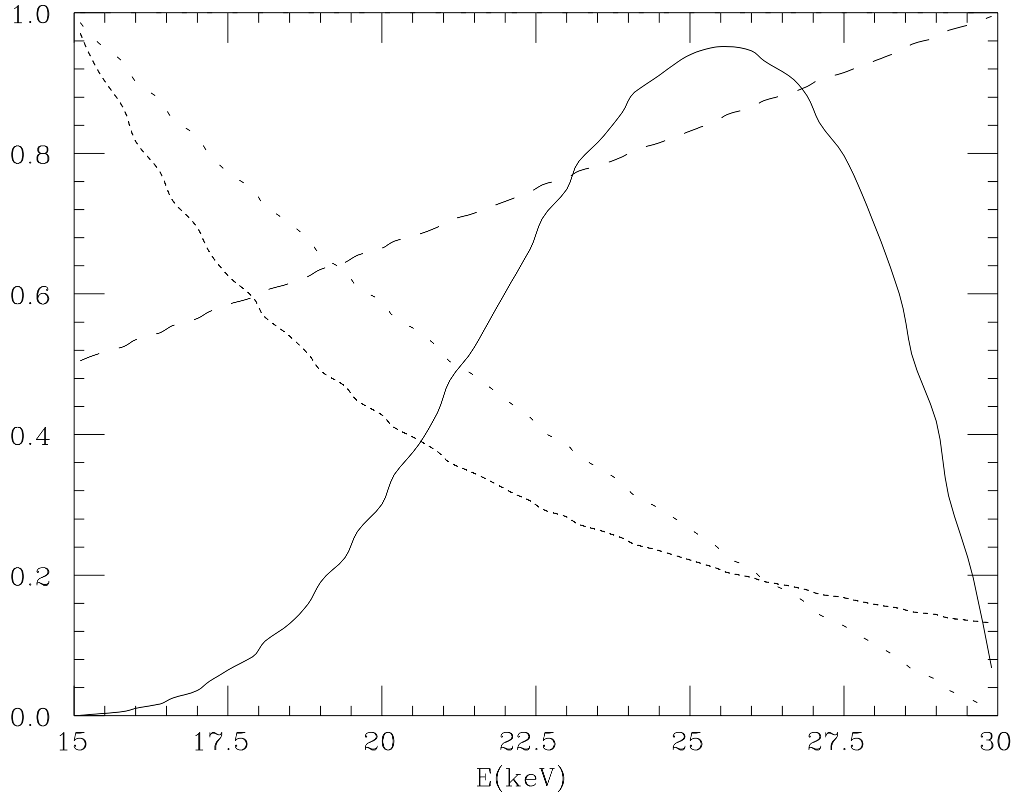

Figure 1: The observed x-ray spectrum (solid). The weighting functions considered: proportional to energy, E (dashed); proportional to σ_T (dotted), proportional to $1 - \log(E/15\mathrm{keV})/\log 2$ (dot-dash)